G-BLUP without inverting the genomic relationship matrix

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Outline

Traditional BLUP

EBV's are obtained by solving Henderson's MME.

$$\begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X'R^{-1}y \\ Z'R^{-1}y \end{bmatrix}$$

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- In G-BLUP: $G^{-1} = (G_0 \bigotimes G_{SNP})^{-1}$

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Alternative formulation of MME

As shown by Henderson (1984), the MME can be rearranged into an unsymmetric system by multiplying the random part with $G = G_0 \bigotimes G$

$$\begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ GZ'R^{-1}X & G(Z'R^{-1}Z + G^{-1}) \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X'R^{-1}y \\ GZ'R^{-1}y \end{bmatrix}$$

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Simplification:

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G do not need to be positive definite

Solving strategy

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Strategy for solving the unsymmetric MME

```
Setup LHS and RHS
if enough memory to hold an factorize the system then
    Solve by direct methods
    (Can be performed by mulitcore LAPACK subroutines)
else
    Use iterative methods
    (Gauss-Seidel, PCG, MINRES, ...)
end if
```

Iterative solving strategy

Rearranging the unsymmetric system as:

$$\begin{bmatrix} X'R^{-1}X & 0 \\ 0 & GZ'R^{-1}Z+I \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X'R^{-1}y - X'R^{-1}Z\hat{u} \\ GZ'R^{-1}y - GZ'R^{-1}X\hat{\beta} \end{bmatrix}$$

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Simplification:

$$\begin{bmatrix} X'R^{-1}X & 0 \\ 0 & GZ'R^{-1}Z + I \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X'R^{-1}(y - Z\hat{u}) \\ GZ'R^{-1}(y - X\hat{\beta}) \end{bmatrix}$$

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Solving strategy

Pseudo code for IOD solver for unsymmetric MME

cd=0
Initiate $\hat{u^0}$ (can be a null vector)
while $cd > \varepsilon$ **do**
$y^{*i} = y - Z\hat{u^{i-1}}$
Compute β^i by solve $X'R^{-1}X\hat{\beta}^i = X'R^{-1}y^{*i}$
(direct or iteratively)
$y^{**i} = y - X\hat{\beta}^i$
Compute $\hat{u^i}$ by solve $(GZ'R^{-1}Z + I)\hat{u^i} = GZ'R^{-1}y^{**i}$
(iteratively)
$cd = \frac{\|u^{i-1} - u^i\|}{\|u^i\|}$
end while

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Status for implementation in the DMU-package

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- IOD solver for unsymmetric MME implemented in DMU5

Test of solvers

Solvers tested on the NAV G-BLUP model for Nordic Red Cattle

Data	DRP's (protein) for 3662 bulls
G matrix	5287 animals

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Symmetric MME	60
Unsymmetric MME	
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Non-genomic part	173
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Solutions are identical

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Conclusions and implications

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- As number of typed animals increases, it might be feasible to form elements in G as they are needed using massive parallel computation (GPU's?)
- Replacing G by H (the One-Step relationship matrix)
  See EAAP presentation by Ødegård et. al